TEMPERATURE DISTRIBUTION IN A WEDGE-SHAPED PRISM

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UDC 536.21

A transient three-dimensional heat-conduction problem is solved for a wedge-shaped prism, subject to varying conditions of heat transfer at the boundary; these conditions are either linearly or exponentially dependent upon time.

Constructional elements in the form of wedge-shaped prisms with arbitrary aperture angles are widely used in engineering practice. In calculating the strength of elementary constructions an analysis of the stressed state of elements of this kind (and particularly the thermally-stressed state arising as a result of transient, nonuniform heating) is of particular significance for this reason.

However, few papers have been published in relation to the transient heat-conduction problem for a wedge-shaped prism, and then only on the assumption of a plane temperature distribution, changes in temperature with respect to the height of the prism not been taken into account. There are hardly any closed analytical solutions for problems of transient heat conduction involving heat-transfer boundary conditions varying with respect to both time and position.

In this paper we shall consider the problem of transient heat conduction for a wedge-shaped prism with an arbitrary aperture angle characterized by heat-transfer boundary conditions of the first kind at the faces of the prism (these varying with respect to both time and coordinates) and heat-transfer boundary conditions of the third kind at the ends of the prism and on its rear surface. For the particular cases of linear and exponential time dependences of the boundary temperatures, the solutions relating to the prism are obtained in closed form.

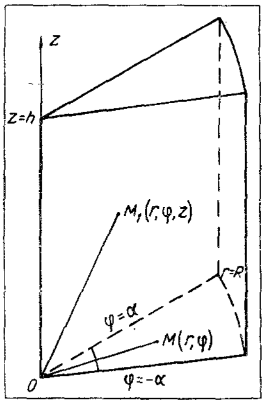

Fig. 1. Computing scheme for the sample, and choice of coordinate system.

Let us consider a wedge-shaped prism of a certain height $h (0 \leq z \leq h)$ and an arbitrary aperture angle $2\alpha (-\alpha \leq \varphi \leq \alpha)$ bounded by a rear surface $r = R$. The choice of the cylindrical coordinate system is illustrated in Fig. 1.

Let us assume that at the initial instant of time $\tau = 0$ the prism is characterized by a specific temperature distribution

$$t(r, \varphi, z, 0) = t_0(r, \varphi, z). \tag{1}$$

On the end surfaces ($z = 0$ and $z = h$) and on the rear surface of the prism ($r = R$) the heat transfer with the surrounding medium is specified by Newton's law with corresponding relative heat-transfer coefficients $\varkappa_1$, $\varkappa_2$, $\varkappa_3$. In order to simplify the calculations we take the temperature of the surrounding medium as zero; this will not restrict the generality of the solution. The boundary conditions of heat transfer may then be written in the following manner [1]:

$$\frac{\partial t}{\partial r} + \varkappa_1 t = 0 \quad \text{for} \quad r = R, \tag{2}$$

$$\frac{\partial t}{\partial z} - \varkappa_2 t = 0 \quad \text{for} \quad z = 0,$$

Institute of Strength Problems, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 23, No. 6, pp. 1100-1106, December, 1972. Original article submitted June 29, 1971.

$$\frac{\partial t}{\partial z} + \varkappa_3 t = 0 \quad \text{for} \quad z = h. \tag{3}$$

The temperatures on the faces of the wedge-shaped prism $\varphi = -\alpha$ and $\varphi = \alpha$ we shall take as being specified functions of the coordinates r and z and the time τ

$$\left.\begin{aligned} t(r, -\alpha, z, \tau) &= t_1(r, z, \tau) \\ t(r, \alpha, z, \tau) &= t_2(r, z, \tau) \end{aligned}\right\}. \tag{4}$$

Thus the problem of determining the unknown temperature distribution $t(r, \varphi, z, \tau)$ in the prism reduces to the integration of the differential equation of transient heat conduction

$$\frac{\partial t}{\partial \tau} = a^2 \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial t}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial z^2} \right)$$
$$(0 \leqslant r \leqslant R, \quad -\alpha \leqslant \varphi \leqslant \alpha, \quad 0 \leqslant z \leqslant h) \tag{5}$$

subject to the initial condition (1) and the boundary condition (2)-(4). The condition of finite temperature as $r \to 0$ is also introduced in the usual way.

In order to solve the problem we set up a system of orthonormalized eigenfunctions depending on the coordinate φ, which we obtain from a solution of the homogeneous Sturm—Liouville problem [2]:

$$\Phi_m(\varphi) = \sqrt{\frac{1}{\alpha}} \cos \frac{(2m+1)\pi}{2\alpha} \varphi$$
$$(m = 0, 1, 2 \ldots). \tag{6}$$

Expressing the unknown temperature-distribution function $t(r, \varphi, z, \tau)$ in the form of an expansion with respect to the system of eigenfunctions (6)

$$t(r, \varphi, z, \tau) = \sum_{m=0}^{\infty} t_m(r, z, \tau) \Phi_m(\varphi), \tag{7}$$

we obtain an inhomogeneous differential equation for the coefficients of the expansion

$$\frac{\partial t_m}{\partial \tau} = a^2 \left(\frac{\partial^2 t_m}{\partial r^2} + \frac{1}{r} \frac{\partial t_m}{\partial r} + \frac{\partial^2 t_m}{\partial z^2} - \frac{\nu^2}{r^2} t_m \right) + F_m, \tag{8}$$

which is to be solved subject to the initial condition

$$t_m(r, z, 0) = (t_m)_0 = \int_{-\alpha}^{\alpha} t_0(r, \varphi, z) \Phi_m(\varphi) d\varphi \tag{9}$$

and the boundary conditions

$$\frac{\partial t_m}{\partial r} + \varkappa_1 t_m = 0 \quad \text{for} \quad r = R, \tag{10}$$

$$\left.\begin{aligned} \frac{\partial t_m}{\partial z} - \varkappa_2 t_m = 0 \quad \text{for} \quad z = 0 \\ \frac{\partial t_m}{\partial z} + \varkappa_3 t_m = 0 \quad \text{for} \quad z = h \end{aligned}\right\}. \tag{11}$$

In Eq. (8) we have introduced the nomenclature

$$\nu = \frac{2m+1}{2\alpha} \pi,$$
$$F_m(r, z, \tau) = (-1)^m \frac{a^2 \nu}{r^2 \sqrt{\alpha}} [t_1(r, z, \tau) + t_2(r, z, \tau)]. \tag{12}$$

Let us apply a finite integral transformation of the Hankel type to the boundary problem (8)-(11) in respect of the variable r; we define this in the following way [3]:

$$T_m(\lambda_k, z, \tau) = \int_0^R r t_m(r, z, \tau) J_\nu(\lambda_k r) dr, \tag{13}$$

where λ_k are the positive roots of the equations

$$\lambda J_\nu'(\lambda R) + \varkappa_1 J_\nu(\lambda R) = 0. \tag{14}$$

According to [3], the equation for inverting the integral transformation (13) takes the form

$$t_m(r, z, \tau) = \frac{2}{R^2} \sum_{k=1}^{\infty} \frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R) + J_\nu'^2(\lambda_k R)} T_m(\lambda_k, z, \tau). \tag{15}$$

For the function $T_m(\lambda_k, z, \tau)$, after applying the transformation (13) to the problem (8)-(11), we obtain the differential equation

$$\frac{\partial T_m}{\partial \tau} = a^2\left(\frac{\partial^2 T_m}{\partial z^2} - \lambda_k^2 T_m\right) + \bar{F}_m, \tag{16}$$

where

$$\bar{F}_m(\lambda_k, z, \tau) = \int_0^R r F_m(r, z, \tau) J_\nu(\lambda_k r) dr$$

with the initial condition

$$T_m(\lambda_k, z, 0) = (T_m)_0 = \int_0^R r (t_m)_0 J_\nu(\lambda_k r) dr \tag{17}$$

and the end conditions

$$\begin{aligned} &\frac{\partial T_m}{\partial z} - \varkappa_2 T_m = 0 \quad \text{for} \quad z = 0, \\ &\frac{\partial T_m}{\partial z} + \varkappa_3 T_m = 0 \quad \text{for} \quad z = h. \end{aligned} \tag{18}$$

To the problem (16)-(18) we now apply an integral transformation with respect to the variable z which was used in [4] for the heat-conduction problem in a cylinder of finite length

$$\bar{T}_m(\lambda_k, \mu_i, \tau) = \int_0^h T_m(\lambda_k, z, \tau) Z(\mu_i z)\, dz, \tag{19}$$

where

$$Z(\mu_i z) = \cos \mu_i z + \frac{\varkappa_2}{\mu_i} \sin \mu_i z. \tag{20}$$

The function $Z(\mu_i z)$ satisfies the equation

$$\frac{d^2 Z}{dz^2} + \mu_i^2 Z = 0 \tag{21}$$

and the conditions

$$\left.\begin{aligned} &\frac{dZ}{dz} + \varkappa_3 Z = 0 \quad \text{for} \quad z = h \\ &\frac{dZ}{dz} - \varkappa_2 Z = 0 \quad \text{for} \quad z = 0 \end{aligned}\right\}, \tag{22}$$

where μ_i are positive roots of the equation

$$\operatorname{tg} \mu h = \frac{\varkappa_2 + \varkappa_3}{\mu^2 - \varkappa_2 \varkappa_3} \mu. \tag{23}$$

The equation for inverting the transformation (19) takes the form

$$T_m(\lambda_k, z, \tau) = \sum_{i=1}^{\infty} A_i \bar{T}_m(\lambda_k, \mu_i, \tau) Z(\mu_i z), \tag{24}$$

where

$$A_i = \left[\int_0^h Z^2(\mu_i z)\, dz\right]^{-1} = \frac{2\mu_i^2 (\mu_i^2 + \varkappa_3^2)}{(\mu_i^2 + \varkappa_2^2)(\mu_i^2 + \varkappa_3^2) h + (\varkappa_2 + \varkappa_3)(\mu_i^2 + \varkappa_2 \varkappa_3)}$$

The function $\bar{T}_m(\lambda_k, \mu_i, \tau)$ satisfies the ordinary differential equation

$$\frac{d\overline{T}_m}{d\tau}=a^2(-\mu_i^2\overline{\overline{T}}_m-\lambda_k^2\overline{T}_m)+\overline{\overline{F}}_m \tag{25}$$

with the initial condition

$$\overline{T}_m(\lambda_k,\ \mu_i,\ 0)=\int\limits_0^h (T_m)_0\, Z(\mu_i z)\, dz. \tag{26}$$

In Eq. (25)

$$\overline{\overline{F}}_m(\lambda_k,\ \mu_i,\ \tau)=\int\limits_0^h \overline{F}_m(\lambda_k,\ z,\ \tau)\, Z(\mu_i z)\, dz. \tag{27}$$

The general solution of Eq. (25) for the condition (26) takes the form

$$\overline{T}_m(\lambda_k,\ \mu_i,\ \tau)=\int\limits_0^\tau \overline{\overline{F}}_m(\lambda_k,\ \mu_i,\ u)\exp[-a^2(\lambda_k^2+\mu_i^2)(\tau-u)]\,du$$
$$+\int\limits_0^h (T_m)_0 Z(\mu_i z)\,dz \exp[-a^2(\mu_i^2+\lambda_k^2)\tau]. \tag{28}$$

Carrying out the inverse transformations in accordance with (24) and (15), we finally obtain the temperature distribution in the wedge-shaped prism in the form:

$$t(r,\ \varphi,\ z,\ \tau)=\frac{2}{R^2\sqrt{\alpha}}\sum_{m=0}^{\infty}\cos\frac{(2m+1)\pi}{2\alpha}\varphi.$$
$$\times\sum_{k=1}^{\infty}\frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R)+J_\nu'^2(\lambda_k R)}\sum_{i=1}^{\infty}A_i\left\{\int\limits_0^\tau \overline{\overline{F}}_m(\lambda_k,\ \mu_i,\ u)\right.$$
$$\times\exp[-a^2(\lambda_k^2+\mu_i^2)(\tau-u)]\,du+\int\limits_0^h (T_m)_0 Z(\mu_i z)\,dz$$
$$\left.\times\exp[-a^2(\lambda_k^2+\mu_i^2)\tau]\right\}\left(\cos\mu_i z+\frac{\varkappa_2}{\mu_i}\sin\mu_i z\right). \tag{29}$$

Using Eq. (29) we may determine the temperature-distribution function in the wedge-shaped prism for any boundary conditions of the type (1)-(4).

It is of considerable practical interest to determine the temperature distributions in a sample of wedge-like profile for the case in which the temperature of the faces varies in accordance with a specific time law. Boundary conditions of this kind may be used for studying the kind of sample heating or cooling conditions actually encountered in experimental work.

Let us consider the particular case in which the temperatures of the sample faces vary linearly with time at a certain constant rate of change v, i.e., we specify the boundary conditions (4) thus:

$$\begin{gathered} t(r,\ -\alpha,\ z,\ \tau)=t_0+v\tau, \\ t(r,\ \alpha,\ z,\ \tau)=t_0+v\tau. \end{gathered} \tag{30}$$

We taken the initial sample temperature as constant

$$t(r,\ \varphi,\ z,\ 0)=t_0=\text{const}. \tag{31}$$

Allowing for (9), (12), (17), and (27), we then obtain the temperature distribution in a wedge-shaped prism subject to the boundary conditions (30) in the following form:

$$t(r,\ \varphi,\ z,\ \tau)=\frac{2}{\alpha R^2}\sum_{m=0}^{\infty}(-1)^m\cos\frac{(2m+1)\pi}{2\alpha}\varphi$$
$$\times\sum_{k=1}^{\infty}\frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R)+J_\nu'^2(\lambda_k R)}\sum_{i=1}^{\infty}A_i\left(\cos\mu_i z+\frac{\varkappa_2}{\mu_i}\sin\mu_i z\right)$$

$$\times\left\{\frac{v}{\mu_i^2+\lambda_k^2}\left[t_0+v\tau-\frac{v}{a^2(\lambda_k^2+\mu_i^2)}-\left(t_0-\frac{v}{a^2(\lambda_k^2+\mu_i^2)}\right)e^{-a^2(\mu_i^2+\lambda_k^2)\tau}\right]\right.$$
$$\times\left[(\nu-2)\lambda_k R J_\nu(\lambda_k R)S_{-2,\nu-1}(\lambda_k R)-\lambda_k R J_{\nu-1}(\lambda_k R)S_{-1,\nu}(\lambda_k R)+\frac{1}{\nu}\right]$$
$$\times\frac{\mu_i\sin\mu_i h-\varkappa_2\cos\mu_i h+\varkappa_2}{\mu_i^2}+\frac{2t_0}{\nu\lambda_k^2R^2}e^{-a^2(\mu_i^2+\lambda_k^2)\tau}$$
$$\left.\times\left[\nu\lambda_k R J_\nu(\lambda_k R)S_{0,\nu-1}(\lambda_k R)-\lambda_k R J_{\nu-1}(\lambda_k R)S_{1,\nu}(\lambda_k R)+\frac{\nu}{\lambda_k^2}\right]\right\}. \quad (32)$$

In expression (32) the coefficient A_i is determined in accordance with (24) while $S_{p,q}(z)$ is a Lommel function [5].

When the temperature on the faces of the prism varies in accordance with an exponential time law with a certain variation coefficient v, i.e., the boundary conditions (4), take the form

$$t(r,\ -\alpha,\ z,\ \tau)=t(r,\ \alpha,\ z,\ \tau)=t_0\exp(-v\tau), \quad (33)$$

after repeating all the foregoing transformations we obtain the temperature distribution of the sample in the following form:

$$t(r,\ \varphi,\ z,\ \tau)=\frac{2}{\alpha R^2}\sum_{m=0}^{\infty}(-1)^m\cos\frac{(2m+1)\pi}{2\alpha}\varphi$$
$$\times\sum_{k=1}^{\infty}\frac{J_\nu(\lambda_k r)}{J_\nu^2(\lambda_k R)+J_\nu'^2(\lambda_k R)}\sum_{i=1}^{\infty}A_i\left(\cos\mu_i z+\frac{\varkappa_2}{\mu_i}\sin\mu_i z\right)$$
$$\times\left\{\left[\frac{a^2vt_0}{a^2(\lambda_k^2+\mu_i^2)-v}\exp(-v\tau)-\frac{a^2vt_0}{a^2(\lambda_k^2+\mu_i^2)-v}\exp[-a^2(\lambda_k^2+\mu_i^2)\tau]\right]\right.$$
$$\times\left[(\nu-2)\lambda_k R J_\nu(\lambda_k R)S_{-2,\nu-1}(\lambda_k R)-\lambda_k R J_{\nu-1}(\lambda_k R)S_{-1,\nu}(\lambda_k R)+\frac{1}{\nu}\right]$$
$$\times\frac{\mu_i\sin\mu_i h-\varkappa_2\cos\mu_i h+\varkappa_2}{\mu_i^2}+\frac{2t_0}{\nu\lambda_k^2R^2}\exp[-a^2(\lambda_k^2+\mu_i^2)\tau]$$
$$\left.\times\left[\nu\lambda_k R J_\nu(\lambda_k R)S_{0,\nu-1}(\lambda_k R)-\lambda_k R J_{\nu-1}(\lambda_k R)S_{1,\nu}(\lambda_k R)+\frac{\nu}{\lambda_k^2}\right]\right\}. \quad (34)$$

The resultant expressions for transient temperature distributions in a wedge-shaped prism constitute series in known, tabulated functions and may be used for numerical calculations of temperature fields in wedge-shaped samples using an electronic computer.

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